Physics GRE Review Fall 2004
Classical Mechanics Problems

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Classical Mechanics Problem Set

These problems are intended to help you review classical mechanics from 8.01 and 8.09, as well as material that may appear on the Physics GRE but is not covered in the standard MIT Physics curriculum.

1. **Energy and Momentum Conservation**
   A mass m slides down a frictionless ramp of height h and length l. At the bottom, the mass collides and sticks to a larger mass M. Find the velocity v at which the two blocks travel after collision.

   $\mathbf{(A)\sqrt{2gh}m/(m+M)}$
   The problem begins with an energy conservation problem followed by a momentum conservation problem. As the first mass slides down the ramp, energy is conserved. Equating the potential energy at the top with the kinetic energy at the bottom gives:

   $$mgh = \frac{1}{2}mv_1^2 \Rightarrow v_1 = \sqrt{2gh}$$

   where v_1 is the speed of the small block when it reaches the bottom of the ramp but before it impacts the larger block. Secondly, we have a completely inelastic collision in which momentum is conserved. Equating momentum before and after the collision gives:

   $$mv_1 = (m+M)v \Rightarrow v = mv_1/(m+M)$$
   $$\Rightarrow v = \sqrt{2gh} * m/(m+M)$$

2. **Ballistic Pendulum**
   A bullet of mass m and velocity v_0 is shot into a wood block of mass M suspend on a string of length l. Find the maximum height the block will reach after collision.

   $\mathbf{(D)(mv_0)^2/[2g(m+M)^2]}$
   This problem has the opposite order of conservation laws as the first problem. First we must consider momentum conservation in the inelastic collision between the bullet and the block:

   $$p = mv_0 \tag{1}$$

where it will be useful to simply use momentum here. The kinetic energy at the point of collision then must equal the potential energy at the highest point:

$$\frac{p^2}{2m_{tot}} = m_{tot}gh$$
$$\Rightarrow h = \frac{p^2}{2gm_{tot}^2}$$
$$\Rightarrow h = \frac{(mv_0)^2}{2g(m+M)^2}$$

3. **Static Equilibrium**
A beam of length l and mass m is supported by a wall-mounted pin at one end and at the other end by a cable attached to the wall a height h above the beam. A man of mass M stands a distance x from the wall on the beam. Find the tension in the cable.

(C)$(\mathbf{m/2 + Mx/l})/(\mathbf{h}/\sqrt{\mathbf{h^2 + l^2}})$ *Note this is an error in the problem set. All the answers had multiplication $*$ with the last term.*
This system is in static equilibrium, which means that the sums of all forces and all torques is equal to zero. To solve this problem we need only consider the sum of torques:

$$\sum \tau = 0$$
$$ml/2 + Mx - T\sin\theta l = 0$$
$$\Rightarrow T = \frac{ml/2 + Mx}{l\sin\theta}$$
$$\Rightarrow T = \frac{m/2 + Mx/l}{h/\sqrt{h^2 + l^2}}$$

4. **Gauss's Law of Gravitation**
A spherical shell with inner radius a and outer radius b has a uniform mass density ρ. Find the gravitational field within the region $a \leq r \leq b$.

(D)$(\mathbf{4\pi/3})\mathbf{G}\rho(\mathbf{r - a^3/r^2})$
To solve this problem, one must use Gauss's Law of Gravitation, choosing a spherical Gaussian surface in the interior of the solid:

$$\oint_S \vec{g}\cdot d\vec{a} = 4\pi G M_{encl} = 4\pi G \int_V \rho dV$$
$$4\pi r^2 g = 4\pi G\rho[\frac{4\pi}{3}r^3 - \frac{4\pi}{3}a^3]$$
$$\Rightarrow g = \frac{4\pi}{3}G\rho[r - \frac{a^3}{r^2}] \qquad (2)$$

5. **Stellar Gravitation**
   Two stars in a binary system orbit one another with a period T at a distance a. Find the mass M of each star.
   (B)$\mathbf{2\pi^2 a^3/GT^2}$
   Simply using Kepler's relation, $4\pi^2 a^3 = GMT^2$, for a single star would give an incorrect answer, but using $M \rightarrow 2M$ does in fact give the correct answer. To see why, let us equate the centripetal force to the gravitational force between the two stars:

$$\begin{aligned} F_C &= F_G \\ M\omega^2 a &= \frac{GMM}{a^2} \\ M(\frac{2\pi}{T})^2 a &= \frac{GMM}{a^2} \\ \Rightarrow M &= 2\pi^2 a^3/GT^2 \end{aligned}$$

6. **Fluid Dynamics**
   An incompressible fluid is stored in a cylindrical tank of radius $R_t = 5m$ and height $h_t = 3m$. Fluid is allowed to escape from a valve of radius $R_v = 5cm$ at the bottom of the tank. Find the rate at which the fluid level in the tank is dropping.
   (C)$\mathbf{7.7 * 10^{-4} m/s}$
   The fact that the liquid is "incompressible" indicates that its density is constant, and let us call that ρ. Since it is exposed to the same pressure at the top and bottom of the tank, we know that pressure is constant as well. This greatly simplifies Bernoulli's equation to an easy form:

$$\begin{aligned} \rho v_1^2/2 + \rho g h_t &= \rho v_2^2/2 \\ \Rightarrow v_1^2 + 2gh_t &= v_2^2 \end{aligned}$$

   where v_1 is the velocity of the liquid at the top of the tank and v_2 is the velocity at the exit valve. This equation alone is not enough to solve the problem. Instead, we need to also use the continuity equation:

$$\begin{aligned} \rho A_1 v_1 &= \rho A_2 v_2 \\ \Rightarrow \pi R_t^2 v_1 &= \pi R_v^2 v_2 \\ \Rightarrow v_2 &= (R_t/R_v)^2 v_1 \end{aligned}$$

   Substituting this back into the first equation with the given values gives:

$$\begin{aligned} v_1^2 + 2gh &= v_1^2 * 10^8 \\ \Rightarrow v_1 &\approx \sqrt{2gh} * 10^{-4} \\ &= \sqrt{19.8 * 3} * 10^{-4} \\ &= 7.7 * 10^{-4} m/s \end{aligned}$$

7. **Viscous Drag**

A sphere of radius $a = 0.03cm$ with a Reynolds number of $R_e = 0.5$ falls through a fluid with kinematic viscosity $\nu = 0.12cm^2/s$ and density $\rho = 1.8 * 10^{-3}g/cm^3$. Find the drag force on the sphere.

(E)$\mathbf{6.79 * 10^{-10}N}$

To solve this problem one must utilize the viscous drag equation:

$$\vec{F}_{drag} = \frac{12\pi a^2 \rho v^2 \hat{v}}{R_e} \tag{3}$$

along with the Reynolds number relation for a sphere: $R_e = 2va/\nu$. This yields the familiar simplification:

$$\begin{aligned} \vec{F}_{drag} &= 3\pi\rho R_e \nu^2 \\ &= 3\pi(1.8 * 10^{-3}g/cm^3)(0.5)/(\nu = 0.12cm^2/s)^2 \\ &= 6.79 * 10^{-10}N \end{aligned}$$

The main point of this problem is to test whether you know the equations to solve the problem. Since calculators are not allowed, the point of this problem is to use order of magnitude calculations to elminate the wrong answers.

8. **Lagrangian Mechanics**

Find the Lagrangian that gives rise to the following force law: $F = m\ddot{x} = -\alpha(3x^3 + 2x)$ with $\alpha > 0$

(D)$\mathbf{\frac{1}{2}m\dot{x}^2 - \frac{3\alpha}{4}x^4 - \alpha x^2}$

Problems like this might be included in the GRE to ensure that you have familiarity with Lagrangian formalism. To find the Lagrangian, solve for the potential from the force law above, using:

$$\vec{F} = -\vec{\nabla}U \tag{4}$$

and substituting the result into the Lagrangian equation:

$$L = T - U \tag{5}$$

which gives the above answer (the rest is simply calculus).

9. **Hamiltonian Mechanics**

Find the Hamiltonian that gives rise to the following force law: $F = dp_x/dt = -\beta(x^5 + 2x^3)$, with $\beta > 0$

(C)$\mathbf{\frac{p_x^2}{2m} + \beta(\frac{x^6}{6} + \frac{x^4}{2})}$

Again, you must find the potential function that gives rise to the given force law and replace it in the Hamiltonian equation:

$$H = T + U \tag{6}$$

Fortunately the Hamiltonian equations of motion also give the standard relation:

$$\vec{F} = -\vec{\nabla} U \tag{7}$$

which solves to the above answer. You may not encounter a problem with answer like this in the real GRE, where two answers are technically correct but with different notation. The point here was to make sure you recall the the Hamiltonian is a function of positions and *their associated momenta*, not their time derivatives.